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Electrodynamics in Quantum Space-Time

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It is shown that introducing quantum space-time into physics leads to a certain fictitious "gravitational" field with the "metric tensor" $\hat{g}_{\mu\nu}(z)$. This formalism allows us to reformulate Maxwell's equations in quantum space-time in accordance with the general covariance principle. A transformation method connecting the electromagnetic field tensor $\hat{F}^{\mu\nu}(z)$ to the usual one $F^{\mu\nu}(x)$ for the quantum system of reference is given. The electromagnetic force acting on the charged particle in quantum space-time is also defined and its equation of motion is investigated for a concrete case. In our scheme, a gauge-invariant description of electromagnetic processes in quantum space-time at small distances is achieved by using gauge transformation over the whole space-time on the large scale.

1. INTRODUCTION

In earlier work (Namsrai, 1986, 1987) within the framework of quantum space-time with noncommuting coordinates

$$x^{\mu} \Rightarrow \hat{z}^{\mu} = x^{\mu} + L\Pi^{\mu}(x) \tag{1}$$

where $\Pi^{\mu}(x)$ is a matrix function of x^{μ} and L is the fundamental length, we have considered gravitational effects and have presented a mathematical method concerning tensor analysis. The aim of this paper is to generalize that formalism to electrodynamics and to study the equation of motion of charged particles in quantum space-time. It turns out that the introduction of quantum space-time into physics on the level of the affine connection gives rise to an additional fictitious "gravitational" field with metric tensor $\hat{g}_{\mu\nu}(z)$. Such a formalism allows us to choose a way for the construction of electrodynamics by using the prescription of the general covariance principle reformulated in the case of the quantum system of reference [see Namsrai (1986) for details]. With this idea, we should first write down equations in

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the same form as in the special theory of relativity and next explain how to change every quantity entering into these equations under an arbitrary transformation (including a quantum one) of coordinates. The equations obtained will be general, covariant and correct in the absence of quantum properties of space-time ($L \to 0$) and therefore preserve their form in an arbitrary fictitious "gravitational" field under the condition that the system considered is sufficiently small with respect to the scale of the fields.

It should be noted that the reformulation of the general covariance principle is possible only up to terms of order L^2 and that the tensor nature of the electromagnetic field tensor $\hat{F}^{\mu\nu}(z)$ in quantum space-time is preserved at this level of accuracy. By analogy with the usual transformation case, in the quantum system of reference we find a simple connection between $\hat{F}^{\mu\nu}(z)$ and the usual $F^{\mu\nu}(x)$, which ensures gauge invariance of generalized electrodynamics on the whole space-time obtained by averaging over the quantum one. In our case, a strict order of multipliers and a definite arrangement of tensor indices for any physical quantities are important. For example, the order of the product of operators defining the electromagnetic force acting on a charged particle may be chosen in such a way that this generalized averaged force in quantum space-time coincides exactly with the usual one.

This paper is organized as follows. In Section 2 we introduce quantum space-time with coordinates (1) leading to an additional fictitious "gravitational" field with metric tensor $\hat{g}_{\mu\nu}(z)$. Section 3 deals with the electromagnetic field tensor $\hat{F}^{\mu\nu}(z)$ and its transformation law with respect to the quantum system of reference. Here the connection between the averaged $\langle \hat{F}^{\mu\nu}(z) \rangle$ and the usual electric **E** and magnetic **H** field strengths is also established. In Section 4 we generalize the Maxwell equations in quantum space-time by using the definition of the covariant derivative given by the affine connection

$$\hat{\Gamma}^{\lambda}_{\mu\nu}(z) = \frac{\partial^2 x^\rho}{\partial z^\mu \, \partial z^\nu} \frac{\partial z^\lambda}{\partial x^\rho}$$

in the fictitious "gravitational" field. Section 5 is devoted to the definition of the electromagnetic force acting on a charged particle in a quantum space-time and to the study of its equation of motion for a concrete case.

2. QUANTUM SPACE-TIME AND FICTITIOUS "GRAVITATIONAL" FIELD

Now we show that introducing quantum space-time with coordinates (1) leads to a fictitious "gravitational" field. Our construction is based on the affine connection method formulated by means of the principle of constancy of the light velocity in different coordinate systems. In the language of proper time it means that $d\tau^2 =$ invariant. For the usual four-dimensional Minkowski space the latter is given by

$$d\tau^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2}$$

where

$$\eta_{\alpha\beta} = \begin{cases} -1 & \alpha = \beta = 1, 2, 3 \\ 1 & \alpha = \beta = 0 \\ 0 & \alpha \neq \beta \end{cases}$$

is the Minkowski metric.

Before defining proper time $d\tau^2$ in quantum space-time we should note that in this case there exist two types of differentiation: left-hand and right-hand, leading to different results [see Namsrai (1987) for details]. For example,

$$\frac{\vec{\partial}}{\partial z^\mu} f(z) = \frac{\partial x^q}{\partial z^\mu} \frac{\partial f(z)}{\partial x^q} \neq f(z) \frac{\overleftarrow{\partial}}{\partial z^\mu} = \frac{\partial f(z)}{\partial x^q} \frac{\partial x^q}{\partial z^\mu}$$

In particular, for the Jacobian type of transformation matrices

$$\frac{\partial x^\alpha}{\partial z^\mu} \frac{\partial z^\lambda}{\partial x^\alpha} \neq \frac{\partial z^\lambda}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial z^\mu}$$

So, for the concrete form of transformation (1), we have

$$\begin{aligned} \frac{\partial x^\rho}{\partial z^\mu} dz^\mu &= dx^\rho + L^2 I^{\delta\rho}_{\nu\delta}\, dx^\nu \\ dz^\mu \frac{\partial x^\rho}{\partial z^\mu} &= dx^\rho \end{aligned} \tag{3a}$$

where

$$I^{\delta\rho}_{\nu\delta} = \frac{\partial \Pi^\delta}{\partial x^\nu} \frac{\partial \Pi^\rho}{\partial x^\delta} - \frac{\partial \Pi^\rho}{\partial x^\delta} \frac{\partial \Pi^\delta}{\partial x^\nu} = -I^{\rho\delta}_{\delta\nu} \tag{3b}$$

Taking into account the equality (3a), it is natural to write (2) in the following invariant form:

$$d\tau'^2 = dz^\nu\, dz^\mu\, \hat{g}_{\mu\nu}(z) \tag{4}$$

where

$$\hat{g}_{\mu\nu}(z) = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial z^\mu} \frac{\partial x^\beta}{\partial z^\nu} \tag{5}$$

is the generalized "metric" tensor in quantum space-time. Here it should be noted that in definition (4) a strict order of multipliers and a definite arrangement of tensor indices for $\hat{g}_{\mu\nu}(z)$ are important in the sense that any other kinds of expressions of the type of (4) break its invariant form. Thus, we see that a requirement of the proper time invariance principle in quantum space-time gives rise inevitably to the appearance of an additional fictitious "gravitational" field or equivalently a curved space-time with metric tensor (5).

Now we explain how to change the equations of motion of a particle in the presence of this additional "gravitational" field. The equation of motion obtained by using the concept of quantum space for the nonrelativistic case and generalized by us (Namsrai, 1986) to the relativistic case has the form

$$\frac{d^2x^\alpha}{d\tau^2}=\frac{f_q^\alpha(x)}{mc} \tag{6}$$

where

$$f_q^\alpha(x)=\left\{\frac{\mathbf{f}_q}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2},\frac{\mathbf{f}_q\cdot\mathbf{v}}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right\}$$

and $\mathbf{f}_q$ is some quantum force. In the particular case when the matrix function $\Pi^i(x)$ in (1) for the spatial quantum variable z^i does not depend on the time, $\mathbf{f}_q$ is determined by the formula

$$f_q^n(x)=-\frac{mL^2}{2}\left(\frac{\partial\Pi^i}{\partial x^n}\frac{\partial^2\Pi^i}{\partial x^j\partial x^m}v^jv^m+\frac{\partial^2\Pi^i}{\partial x^j\,\partial x^m}v^jv^m\frac{\partial\Pi^i}{\partial x^n}\right.$$
$$\left.+\frac{\partial\Pi^i}{\partial x^n}\frac{\partial\Pi^i}{\partial x^j}\dot{v}^j+\frac{\partial\Pi^i}{\partial x^j}\frac{\partial\Pi^i}{\partial x^n}\dot{v}^j\right) \tag{7}$$

$(n=1,2,3)$.

Further, we assume that in the fictitious "gravitational" field given by the metric (5) the coordinates x^α are functions of variables z^μ and equation (6) takes the form

$$\frac{d^2z^\lambda}{d\tau^2}+\frac{dz^\mu}{d\tau}\frac{dz^\mu}{d\tau}\hat{\Gamma}^\lambda_{\nu\mu}(z)=\frac{1}{mc}f^\lambda(z) \tag{8}$$

where

$$f^\lambda(z)=\frac{\partial z^\lambda}{\partial x^\rho}f^\rho(z)\sim f^\lambda(x)$$

in according with definition (7); the L^2 term has been presented already. The expression

$$\hat{\Gamma}^{\lambda}_{\nu\mu}(z) = \frac{\partial^2 x^\alpha}{\partial z^\nu \, \partial z^\mu} \frac{\partial z^\lambda}{\partial x^\alpha} \tag{9}$$

generalizes the definition of the affine connection $\Gamma^{\lambda}_{\nu\mu}(x)$ for the usual theory. In the given case, $\Gamma^{\lambda}_{\nu\mu}(x) = 0$, since we are not considering a real gravitational field. However, $\hat{\Gamma}^{\lambda}_{\nu\mu}(z)$ is not equal to zero in our case. Indeed, by using (1) and taking into account the expression

$$\frac{\partial x^\delta}{\partial z^\nu} = \delta^\delta_\nu - L\frac{\partial \Pi^\delta}{\partial x^\nu} + L^2 \frac{\partial \Pi^\alpha}{\partial x^\nu} \frac{\partial \Pi^\delta}{\partial x^\alpha} + O(L^3)$$

for an inverse operation $\partial x^\delta/\partial z^\nu$ with respect to $\partial z^\mu/\partial x^\delta$, we have

$$\hat{\Gamma}^{\lambda}_{\mu\nu}(z) = -L\frac{\partial^2 \Pi^\lambda}{\partial x^\nu \, \partial x^\mu} + L^2\left(\frac{\partial \Pi^\delta}{\partial x^\nu} \frac{\partial^2 \Pi^\lambda}{\partial x^\mu \, \partial x^\delta} + \frac{\partial \Pi^\delta}{\partial x^\mu} \frac{\partial^2 \Pi^\lambda}{\partial x^\nu \, \partial x^\delta}\right) \tag{10}$$

Equations (6) and (8) should be equivalent because they have been obtained in different ways under the same assumption about the quantum nature of space-time at small distances. Indeed, it is easily verified that with definition (9) and formulas (10) and (11)

$$\frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \hat{\Gamma}^{\lambda}_{\nu\mu}(z) \equiv 0$$

so that the averaged equation (8) coincides explicitly with (6). Thus, the additional "gravitational" field with metric tensor (5) and affine connection (9) does not change the particle's equation of motion obtained in previous work (Namsrai, 1986, 1987), as expected. Nevertheless, the idea of a fictitious "gravitational" field arising from introducing quantum space-time turns out to be useful for reformulating electrodynamics within this point of view. Now we go on to this problem.

3. ELECTROMAGNETIC FIELD TENSOR $\hat{F}^{\mu\nu}(z)$ IN QUANTUM SPACE-TIME

To reformulate electrodynamics in quantum space-time, a transformation law of the electromagnetic field tensor $\hat{F}^{\mu\nu}(z)$ under the "quantum" passage (1) needs to be formulated. Here in accordance with the correspondence principle we require that the connection between the components of $\hat{F}^{\mu\nu}(z)$ and the generalized electric $\hat{E}(z)$ and magnetic $\hat{H}(z)$ field strengths remains valid in quantum space-time, i.e.,

$$\begin{gathered}\hat{F}^{12}=\hat{H}_3, \qquad \hat{F}^{23}=\hat{H}_1, \qquad \hat{F}^{31}=\hat{H}_2 \\ \hat{F}^{01}=\hat{E}_1, \qquad \hat{F}^{02}=\hat{E}_2, \qquad \hat{F}^{03}=\hat{E}_3 \\ \hat{F}^{\alpha\beta}=-F^{\beta\alpha}\end{gathered} \tag{11}$$

Our next problem is to define the antisymmetric tensor $\hat{F}^{\mu\nu}(z)$ transformed by means of the matrix $\partial z^\rho/\partial x^\delta$. A simple form of this tensor is

$$\hat{F}^{\mu\nu}(z)=\frac{1}{2}\left[\frac{\partial z^\mu}{\partial x^\rho}\frac{\partial z^\nu}{\partial x^\delta}F^{\rho\delta}(x)+\frac{\partial z^\nu}{\partial x^\delta}\frac{\partial z^\mu}{\partial x^\delta}\bigg)F^{\delta\rho}(x)\right] \tag{12}$$

or, using the concrete representation (1), we get

$$\begin{aligned}\hat{F}^{\mu\nu}(z)=F^{\mu\nu}(x)&+L\left(\frac{\partial\Pi^\mu(x)}{\partial x^\rho}F^{\rho\nu}+\frac{\partial\Pi^\nu(x)}{\partial x^\delta}F^{\mu\delta}\right)\\ &+\frac{L^2}{2}F^{\delta\rho}(x)\left(\frac{\partial\Pi^\nu(x)}{\partial x^\rho}\frac{\partial\Pi^\mu(x)}{\partial x^\delta}+\frac{\partial\Pi^\mu(x)}{\partial x^\delta}\frac{\partial\Pi^\nu(x)}{\partial x^\rho}\right)\end{aligned} \tag{13}$$

To average the latter expression, we must know the explicit form of the matrix function $\Pi^\nu(x)$. In the tetrad representation

$$\Pi^\mu(x)=e^\mu_a(x)\cdot\gamma^a$$

where

$$e^\mu_a(x)=\begin{pmatrix} e^0_0=1 & e^0_1=0 & e^0_2=0 & e^0_3=0 \\ e^1_0=0 & e^1_1=x/r & e^1_2=y/r & e^1_3=z/r \\ e^2_0=0 & e^2_1=xz/r\rho & e^2_2=zy/r\rho & e^2_3=-\rho/r \\ e^3_0=0 & e^3_1=-y/\rho & e^3_2=x/\rho & e^3_3=0\end{pmatrix} \tag{14}$$

$$r=(x^2+y^2+z^2)^{1/2}, \qquad \rho=(x^2+y^2)^{1/2}$$

(γ^a are Dirac matrices) the averaged values of (11) or (13) are expressed by the usual quantities $\mathbf{E}$ and $\mathbf{H}$:

$$\begin{aligned}\mathcal{H}_1&=\langle\hat{H}_1\rangle=\langle\hat{F}^{23}\rangle=H_1+L^2\mathcal{D}\\ \mathcal{H}_2&=\langle\hat{H}_2\rangle=\langle\hat{F}^{31}\rangle=H_2+L^2(z/\rho)\mathcal{D}\\ \mathcal{H}_3&=\langle\hat{H}_3\rangle=\langle\hat{F}^{12}\rangle=H_3\\ \mathcal{E}^i&=\langle\hat{E}^i\rangle=\langle\hat{F}^{0i}\rangle=E^i\end{aligned} \tag{15}$$

Here

$$\mathcal{D}=\frac{1}{r^3}(\mathbf{Hr}+\mathbf{EP}), \qquad \mathbf{p}=[\mathbf{r}\times\mathbf{v}]$$

The problem of the unique construction of the covariant tensor $\hat{F}_{\mu\nu}(z)$ encounters a difficulty connected with the uncertainty in the choice of the order of multipliers and a definite arrangement of tensor indices in its definition. For example, rewriting (12) in the form

$$\hat{F}^{\mu\nu}(z)=\hat{F}_1^{\mu\nu}(z)+\hat{F}_2^{\nu\mu}(z)=\frac{1}{2}\frac{\partial z^\mu}{\partial x^\rho}\frac{\partial z^\nu}{\partial x^\delta}F^{\rho\delta}_{(x)}+\frac{1}{2}\frac{\partial z^\nu}{\partial x^\rho}\frac{\partial z^\mu}{\partial x^\delta}F^{\delta\rho}(x)$$

we get the following results:

Case (i):

$$\begin{aligned}\hat{F}_{\lambda\varkappa}(z)&=\hat{g}_{\lambda\mu}(z)\hat{F}_1^{\mu\nu}(z)\hat{g}_{\nu\varkappa}(z)+\hat{g}_{\varkappa\nu}(z)\hat{F}_2^{\nu\mu}(z)\hat{g}_{\mu\lambda}(z)\\&=\frac{\partial x^\alpha}{\partial z^\lambda}\frac{\partial x^\beta}{\partial z^\varkappa}F_{\alpha\beta}+\frac{1}{2}L^2F_{\beta\alpha}I^{\alpha\beta}_{\varkappa\lambda}+\frac{1}{2}L^2F^{\rho\delta}I^{\gamma\beta}_{\rho\gamma}\\&\quad\times(\eta_{\lambda\beta}\eta_{\delta\varkappa}-\eta_{\varkappa\beta}\eta_{\delta\lambda})\end{aligned}\tag{16a}$$

Case (ii):

$$\begin{aligned}\hat{F}_{\lambda\varkappa}(z)&=F_1^{\mu\nu}(z)\hat{g}_{\nu\varkappa}(z)\hat{g}_{\mu\lambda}(z)+\hat{F}_2^{\nu\mu}(z)\hat{g}_{\mu\nu}(z)\hat{g}_{\nu\varkappa}(z)\\&=F_{\alpha\beta}\frac{\partial x^\alpha}{\partial z^\lambda}\frac{\partial x^\beta}{\partial z^\varkappa}+\frac{1}{2}L^2F_{\beta\alpha}I^{\alpha\beta}_{\varkappa\lambda}+\frac{1}{2}L^2\eta_{\delta\beta}F^{\mu\delta}\\&\quad\times(I^{\beta\alpha}_{\varkappa\mu}\eta_{\alpha\lambda}-\eta_{\alpha\varkappa}I^{\beta\alpha}_{\lambda\mu})\end{aligned}\tag{16b}$$

Here, by construction, $\hat{F}_{\lambda\varkappa}(z)=-\hat{F}_{\varkappa\lambda}(z)$ for both cases, and $I^{\alpha\beta}_{\varkappa\lambda}$ is given by expression (3b).

Moreover, it is possible to generalize the usual definition

$$F_{\mu\nu}(x)=\frac{\partial A_\mu(x)}{\partial x^\nu}-\frac{\partial A_\nu(x)}{\partial x^\mu}$$

in accordance with the correspondence principle for the fictitious "gravitational" field:

$$\hat{F}_{\mu\nu}(z)=A_{\mu;\nu}-A_{\nu;\mu}+[\hat{\Gamma}^\rho_{\nu\mu}(z)-\tilde{\Gamma}^\rho_{\mu\nu}(z)]A_\rho(z)\tag{17}$$

This gives the following result:

Case (iii):

$$\hat{F}_{\mu\nu}(z)=\frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial x^\beta}{\partial z^\nu}F_{\alpha\beta}-L^2I^{\rho\delta}_{\nu\mu}\frac{\partial A_\delta(x)}{\partial x^\rho}\tag{16c}$$

where we have used the identity

$$\frac{\partial^2x^\rho}{\partial z^\mu\,\partial z^\nu}=\hat{\Gamma}^\alpha_{\mu\nu}(z)\frac{\partial x^\rho}{\partial z^\alpha}$$

since

$$\frac{\partial z^{\lambda}}{\partial x^{\alpha}}\frac{\partial x^{\rho}}{\partial z^{\lambda}}=\delta_{\alpha}^{\rho}$$

In definition (17), $A_{\mu}(z)_{;\nu}$ denotes the covariant differentiation (Namsrai, 1986)

$$\hat{A}_{\mu}(z)_{;v}=\frac{\partial \hat{A}_{\mu}(z)}{\partial z^{\nu}}-\hat{\Gamma}_{\nu\mu}^{\lambda}(z)\hat{A}_{\lambda}(z)$$

determined by using the affine connection (9).

It should be noted that in expressions (16a)-(16c) the terms $(\partial x^{\alpha}/\partial z^{\lambda})(\partial x^{\beta}/\partial z^{\varkappa})F_{\alpha\beta}(x)$ breaking the tensor structure do not contribute to any observable physical processes, since after averaging, these terms become zero at our level of accuracy (here taking into account quantities of the order of L^2 terms).

Finally, we note that a gauge transformation of the electromagnetic field tensor $\hat{F}^{\mu\nu}(z)$ should be studied on the whole space-time obtained by using the averaging procedure over the quantum space-time, i.e., the usual tensor $F^{\mu\nu}(x)$ and at the same time $\hat{F}^{\mu\nu}(z)$ by construction are invariant under the gauge transformation

$$A^{\mu}(x)\Rightarrow A'^{\mu}(x)=A^{\mu}(x)+\partial f/\partial x^{\mu}$$

where $f(x)$ is an arbitrary function of coordinates x^{ν}.

4. THE MAXWELL EQUATIONS IN QUANTUM SPACE-TIME

In order to generalize the Maxwell equations to quantum space-time, we apply the general covariance principle to a fictitious "gravitational" field arising from the idea of the quantum character of space-time at small distances. For this purpose, we recall that in the absence of the quantum property of space-time or, equivalently, of the fictitious "gravitational" field, Maxwell's electrodynamic equations are

$$\frac{\partial}{\partial x^{\alpha}}F^{\alpha\beta}=-J^{\beta} \tag{18}$$

$$\frac{\partial}{\partial x^{\alpha}}F_{\beta\gamma}+\frac{\partial}{\partial x^{\beta}}F_{\gamma\alpha}+\frac{\partial}{\partial x^{\gamma}}F_{\alpha\beta}=0 \tag{19}$$

where J^{β} is the four-vector $(\mathbf{J}, \varepsilon)$ and $F^{\alpha\beta}(x)$ is the usual electromagnetic field tensor; $F^{12}=H_3$, $F^{01}=E_1$, etc. (see above). Further, we find $\hat{F}^{\mu\nu}(z)$ and $\hat{J}^{\mu}(z)$ in arbitrary quantum coordinates, which lead to $F^{\alpha\beta}(x)$ and $J^{\beta}(x)$ in quasilocal inertial tensors (determined above) under a quantum transformation of coordinates. Then, one can reduce equations (18) and (19) to general covariant ones by changing all derivatives to covariant ones:

$$\hat{F}^{\mu\nu}(z)_{;\nu} = -\hat{J}^{\mu}(z) \tag{20}$$

$$\hat{F}_{\mu\nu}(z)_{;\lambda} + \hat{F}_{\nu\lambda}(z)_{;\mu} + \hat{F}_{\lambda\mu}(z)_{;\nu} = 0 \tag{21}$$

where

$$\hat{J}^{\mu}(z) = \frac{\partial z^{\mu}}{\partial x^{\rho}} J^{\rho}(x)$$

Now we define covariant differentiation of $\hat{F}^{\mu\nu}(z)$ by using affine connection (9) (Namsrai, 1987). Denote $\hat{T}^{\mu\nu}(z) = \hat{U}^{\mu}(z)\hat{V}^{\nu}(z)$; then, by definition,

$$\begin{aligned}
[\hat{T}^{\mu\nu}(z)]^{z}_{;\lambda} &\overset{Df}{=} \frac{\partial x^{\rho}}{\partial z^{\lambda}} [\hat{T}^{\mu\nu}(z)]^{x}_{;\rho} \\
&= \frac{\partial x^{\rho}}{\partial z^{\lambda}} \{[\hat{U}^{\mu}(z)]^{x}_{;\rho} \hat{V}^{\nu}(z) + \hat{U}^{\mu}(z)[\hat{V}^{\nu}(z)]^{x}_{;\rho}\} \\
&= \hat{U}^{\mu}(z)_{;\lambda} \hat{V}^{\nu}(z) + \frac{\partial x^{\rho}}{\partial z^{\lambda}} \hat{U}^{\mu}(z)[\hat{V}^{\nu}(z)]^{x}_{;\rho}
\end{aligned}$$

where the symbols $[\cdots]^{z}_{;\lambda}$ and $[\cdots]^{x}_{;\lambda}$ mean covariant differentiation with respect to variables z^{λ} and x^{λ}, respectively. Next, the commutator $[\partial x^{\rho}/\partial z^{\lambda}, \hat{U}^{\mu}(z)]_{-}$ needs to be defined. Taking into account $\hat{U}^{\mu}(z) = (\partial z^{\mu}/\partial x^{\tau}) U^{\tau}(x)$, we get

$$\left[\frac{\partial x^{\rho}}{\partial z^{\lambda}}, \hat{U}^{\mu}(z)\right]_{-} = L^2 I^{\mu\rho}_{\tau\lambda} U^{\tau}(x)$$

Therefore,

$$[\hat{T}^{\mu\nu}(z)]^{z}_{;\lambda} = \hat{U}^{\mu}(z)_{;\lambda} \hat{V}^{\nu}(z) + \hat{U}^{\mu}(z)\hat{V}^{\nu}(z)_{;\lambda} + L^2 I^{\mu\rho}_{\tau\lambda} \hat{U}^{\tau}_{(x)}[\hat{V}^{\nu}(z)]^{x}_{;\rho}$$

where

$$\hat{U}^{\mu}(z)_{;\lambda} = \frac{\partial \hat{U}^{\mu}(z)}{\partial z^{\lambda}} + \hat{U}^{\varkappa}(z)\hat{\Gamma}^{\mu}_{\varkappa\lambda}(z)$$

Further, using the equality

$$\frac{\partial}{\partial z^{\lambda}}(\hat{T}^{\mu\nu}(z)) = \frac{\partial \hat{U}^{\mu}(z)}{\partial z^{\lambda}} \hat{V}^{\nu}(z) + \hat{U}^{\mu}(z) \frac{\partial \hat{V}^{\nu}(z)}{\partial z^{\lambda}} + L^2 I^{\mu\rho}_{\tau\lambda} \hat{U}^{\tau}(z) \frac{\partial \hat{V}^{\nu}(z)}{\partial x^{\rho}}$$

we obtain finally

$$\begin{aligned}
\hat{T}^{\mu\nu}(z)_{;\lambda} &= \frac{\partial \hat{T}^{\mu\nu}(z)}{\partial z^{\lambda}} + \hat{T}^{\varkappa\nu}(z)\hat{\Gamma}^{\mu}_{\varkappa\lambda}(z) + \hat{T}^{\mu\varkappa}(z)\hat{\Gamma}^{\nu}_{\varkappa\lambda}(z) \\
&\quad + L^2 T^{\varkappa\delta}(x)[N^{\nu\mu}_{\delta,\varkappa\lambda} + \Gamma^{\nu}_{\rho\delta}(x) I^{\mu\rho}_{\varkappa\lambda}]
\end{aligned} \tag{22}$$

where we have used the commutator

$$[\hat{\Gamma}^{\mu}_{\varkappa\lambda}(z), \hat{V}^{\nu}(z)]_{-} = L^2 N^{\nu\mu}_{\tau,\varkappa\lambda} V^{\tau}(x)$$

and the approximation

$$L^2 I^{\mu\rho}_{\tau\lambda}\, U^\tau(x)\left\{[\hat{V}^\nu(z)]^\varkappa_{;\rho}-\frac{\partial \hat{V}^\nu(z)}{\partial x^\rho}\right\}\approx L^2 I^{\mu\rho}_{\tau\lambda}\, U^\tau(x) V^\varkappa(x)\Gamma^\nu_{\rho\varkappa}(x)$$

Here

$$N^{\nu\mu}_{\tau,\varkappa\lambda}=\frac{\partial\Pi^\nu}{\partial x^\tau}\frac{\partial^2\Pi^\mu}{\partial x^\varkappa\,\partial x^\lambda}-\frac{\partial^2\Pi^\mu}{\partial x^\varkappa\,\partial x^\lambda}\frac{\partial\Pi^\nu}{\partial x^\tau}$$

$\Gamma^\nu_{\rho\varkappa}(x)$ is the usual affine connection. In our case, we can assume $\Gamma^\nu_{\rho\varkappa}(x)\equiv 0$, since we have not considered a real gravitational field.

Thus, covariant differentiation of $\hat{F}^{\mu\nu}(z)$ is given by an expression of the type (22). An analogous calculation may be carried out for $\hat{F}_{\mu\nu}(z)$. The result reads (Namsrai, 1987)

$$\begin{aligned}\hat{F}_{\mu\nu}(z)_{;\lambda}=&\frac{\partial \hat{F}_{\mu\nu}(z)}{\partial z^\lambda}-\hat{\Gamma}^\varkappa_{\lambda\mu}(z)\hat{F}_{\varkappa\nu}(z)-\hat{\Gamma}^\varkappa_{\lambda\nu}(z)\hat{F}_{\varkappa\mu}(z)-L^2F_{\tau\varkappa}(x)\\&\times[\Delta^{\tau\varkappa}_{\mu\lambda\nu}-I^{\tau\delta}_{\mu\lambda}\Gamma^\varkappa_{\nu\delta}(x)]\end{aligned}\tag{23}$$

where

$$\Delta^{\delta\rho}_{\mu\lambda\nu}(x)=I^{\delta\alpha}_{\mu\nu}\Gamma^\rho_{\lambda\alpha}(x)+I^{\delta\beta}_{\mu\lambda}\Gamma^\rho_{\beta\nu}(x)+\frac{\partial}{\partial x^\lambda}I^{\delta\rho}_{\mu\nu}$$

Here $\Gamma^\varkappa_{\nu\delta}\equiv 0$ should be assumed as well.

Finding explicit forms of $\hat{F}^{\mu\nu}(z)_{;\lambda}$ and $\hat{F}_{\mu\nu}(z)_{;\lambda}$ by means of formulas (22) and (23), inserting these values into equations (20) and (21), and averaging, we have

$$\frac{\partial F^{\mu\nu}(x)}{\partial x^\mu}=-J^\nu(x)\tag{24}$$

$$\frac{\partial F_{\mu\nu}(x)}{\partial x^\lambda}+\frac{\partial F_{\nu\lambda}(x)}{\partial x^\mu}+\frac{\partial F_{\lambda\mu}(x)}{\partial x^\nu}+L^2(\langle Q_{\mu\nu\lambda}\rangle+\langle Q_{\nu\lambda\mu}\rangle+\langle Q_{\lambda\mu\nu}\rangle)=0\tag{25}$$

where

$$\begin{aligned}Q_{\mu\nu\lambda}=&\frac{\partial\Pi^\rho}{\partial x^\mu}\frac{\partial\Pi^\delta}{\partial x^\rho}\frac{\partial F_{\delta\nu}}{\partial x^\lambda}+\frac{\partial\Pi^\rho}{\partial x^\lambda}\frac{\partial\Pi^\delta}{\partial x^\mu}\frac{\partial F_{\delta\nu}}{\partial x^\rho}\\&+2\left(\frac{\partial^2\Pi^2}{\partial x^\lambda\,\partial x^\nu}\frac{\partial\Pi^\delta}{\partial x^\rho}+\frac{\partial\Pi^\rho}{\partial x^\nu}\frac{\partial^2\Pi^\delta}{\partial x^\lambda\,\partial x^\rho}+\frac{\partial\Pi^\rho}{\partial x^\lambda}\frac{\partial^2\Pi^\delta}{\partial x^\rho\,\partial x^\nu}\right)F_{\mu\delta}\\&+\frac{\partial\Pi^\rho}{\partial x^\nu}\frac{\partial\Pi^\delta}{\partial x^\rho}\frac{\partial F_{\mu\delta}}{\partial x^\lambda}+\frac{\partial\Pi^\rho}{\partial x^\lambda}\frac{\partial\Pi^\delta}{\partial x^\nu}\frac{\partial F_{\mu\delta}}{\partial x^\rho}+\frac{\partial\Pi^\rho}{\partial x^\mu}\frac{\partial\Pi^\delta}{\partial x^\nu}\frac{\partial F_{\rho\delta}}{\partial x^\lambda}+\frac{\partial\Pi^\rho}{\partial x^\lambda}\frac{\partial\Pi^\delta}{\partial x^\rho}\frac{\partial F_{\mu\nu}}{\partial x^\delta}\\&+\left(\frac{\partial\Pi^\rho}{\partial x^\mu}\frac{\partial^2\Pi^\delta}{\partial x^\lambda\,\partial x^\nu}-\frac{\partial^2\Pi^\rho}{\partial x^\lambda\,\partial x^\nu}\frac{\partial\Pi^\delta}{\partial x^\mu}\right)F_{\rho\delta}\end{aligned}$$

In the last terms of the left-hand side of equation (25) the trace of the expressions $Q_{\mu\nu\lambda}, \ldots$ in the tetrad representation should be taken. Equation (24) means that the first two Maxwell equations

$$\nabla \mathbf{E} = \varepsilon, \qquad \nabla \times \mathbf{H} = \partial \mathbf{E}/\partial t + \mathbf{J}$$

do not change their forms in quantum space-time, or, equivalently, the electromagnetic field current $J^{\mu}(x)$ is conserved over the whole space-time continued from the quantum one. On the contrary, in quantum space-time the other two Maxwell equations

$$\nabla \mathbf{H} = 0, \qquad \nabla \times \mathbf{E} = -\partial \mathbf{H}/\partial t$$

are changed in accordance with (25). Investigation of this problem is left to future work.

5. EQUATION OF MOTION OF A CHARGED PARTICLE IN QUANTUM SPACE-TIME

In the special theory of relativity the electromagnetic force acting on a charged particle is given by

$$f^{\alpha}_{\mathrm{EM}} = e\eta_{\beta\gamma} F^{\alpha\beta}(x) \frac{dx^{\gamma}}{d\tau} = eF^{\alpha}_{\gamma} \frac{dx^{\gamma}}{d\tau}$$

From this it follows immediately that in the quantum system of reference the electromagnetic force $\hat{f}^{\mu}(z)$ takes the form

$$\hat{f}^{\alpha}(z) = e\hat{F}^{\alpha\beta}(z)\left(\frac{dz^{\gamma}}{d\tau}\right)\hat{g}_{\gamma\beta}(z) \tag{26}$$

where

$$\hat{F}^{\alpha\beta}(z) = \frac{\partial z^{\alpha}}{\partial x^{\rho}} \frac{\partial z^{\beta}}{\partial x^{\delta}} F^{\rho\delta}(x) + \frac{1}{2} L^2 I^{\beta\alpha}_{\rho\delta} F^{\delta\rho}$$

in accordance with definition (12). Thus, taking into account the identity $(\partial z^{\mu}/\partial x^{\rho})\, \partial x^{\delta}/\partial z^{\mu} = \delta^{\delta}_{\rho}$, we obtain

$$\hat{f}^{\alpha}(z) = f^{\alpha}_{\mathrm{EM}}(x) + \frac{1}{2} L^2 e I^{\beta\alpha}_{\rho\delta} F^{\delta\rho}(x) \frac{dx^{\gamma}}{d\tau} \eta_{\gamma\beta}$$

and therefore the averaged force in quantum space-time is equal to the usual electromagnetic one:

$$\langle \hat{f}^{\alpha}(z) \rangle = f^{\alpha}_{\mathrm{EM}}(\mathrm{x})$$

With the free particle equation (6) or (8) in quantum space-time, the averaged equation of motion for a charged particle reads

$$\frac{d^2x^\alpha}{d\tau^2}=\frac{1}{mc}f^\alpha_{\mathrm{EM}}(x)+\frac{1}{mc}f^\alpha_q(x) \tag{27}$$

In the nonrelativistic limit, equation (27) reduces to the following equation:

$$\frac{d\mathbf{p}}{dt}+\frac{2L^2}{r^2}[\dot{\mathbf{p}}-\mathbf{n}(\mathbf{n}\dot{\mathbf{p}})]-\frac{4L^2m}{r^3}\{(\mathbf{nv})[\mathbf{n}\times(\mathbf{v}\times\mathbf{n})]\}$$

$$=e\mathbf{E}+\frac{e}{c}(\mathbf{v}\times\mathbf{H}),\qquad \mathbf{n}=\frac{\mathbf{r}}{r} \tag{28}$$

for the tetrad field $e^i_j(x)$, $i, j=1,2,3$, obtained from (14) by crossing out the first line and column and regularizing as by Dineykhan and Namsrai (1986). As an example, we consider the magnetic field only and a two-dimensional case. Then the equation of motion in the polar system of coordinates (ρ, φ) takes the form

$$\begin{aligned}(\rho^2+2L^2)\ddot{\varphi}+2\rho\dot{\rho}\dot{\varphi}+\omega\rho\dot{\rho}&=0\\ \ddot{\rho}-\rho\dot{\varphi}^2-\omega\rho\dot{\varphi}&=0\end{aligned} \qquad (29$$

where $\omega=eH/mc$. As a first approximation we use the solution $\rho_0=\text{const}$ and $\varphi_0=-\omega t$ when $L=0$. This solution corresponds to circular motion of the particle in the plane perpendicular to the magnetic field lines, i.e., the (x, y) plane. Further, as a second approximation we seek a solution of (29) in the form $\rho(t)=\rho_0+L^2\rho_1$ and $\varphi(t)=\varphi_0+L^2\varphi_1$. After a simple calculation,

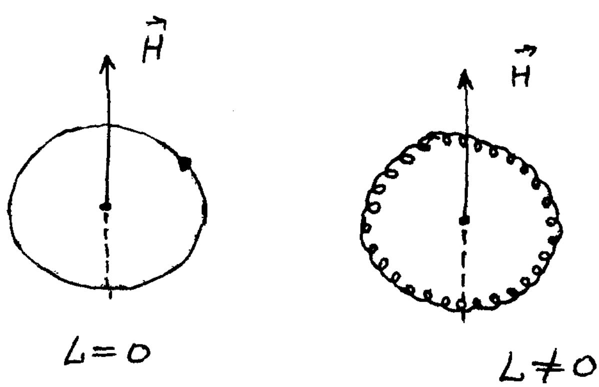

Fig. 1. Illustration of the change of a particle's trajectory due to quantum space-time in an external magnetic field.

we have

$$\rho_1(t) = A \cos \omega t - (\rho_0/\omega) C_1$$
$$\varphi_1(t) = (A/\rho_0) \sin \omega t$$

where A and c_1 are integration constants. Thus, we see that the introduction of a fundamental length (arising from the quantum character of space-time) into the equation of motion leads to a change of the particle's trajectory in an external field. In this particular case, the circle over which the particle moves begins to become twisted, generally speaking, in an arbitrary manner (see Fig. 1 for an illustration).

Finally, we note that in the three-dimensional case, depending on initial conditions, a particle's trajectory is very complicated and behaves like a strange attractor at least in the domain determined by the parameter L in the presence of an external electromagnetic field.

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